On BMO property doe potentials(Function Spaces on Riemann Surfaces)

Gotoh, Yasuhiro

数理解析研究所講究録
571巻 1985年 16-29

On BMO property for potentials

Yasuhiro Gotoh (後藤泰宏)
(Department of Math. Kyoto University)

We can consider the following two BMO spaces naturally for functions on the unit disk D.

(1) BMO space with respect to $dm(z)$ (= $dxdy$).

$$BMO(D,m) = \left\{ f \in L^1_{loc}(D) : \|f\|_m = \sup_B \frac{1}{m(B)} \int_B |f - f(B,m)|\, dm < \infty \right\},$$

where $f(B,m) = \frac{1}{m(B)} \int_B f\, dm$ and the supremum is taken for every disk B ($\overline{B} \subset D$).

$$BMOH(D,m) = BMO(D,m) \cap H(D), \quad BMOA(D,m) = BMO(D,m) \cap A(D).$$

(2) BMO space with respect to $d\lambda(z)$ ($= dxdy/(1-|z|^2)^2$).

$BMO(D,\lambda)$, $BMOH(D,\lambda)$, $BMOA(D,\lambda)$,

are defined similarly.

The following relations are known ([1], [4], [9]).

$$BMO(D,\lambda) \subsetneq BMO(D,m),$$

$$BMOH(D) = BMOH(D,\lambda) \subsetneq BMOH(D,m) = \mathcal{B}_H(D),$$

$$BMOA(D) = BMOA(D,\lambda) \subsetneq BMOA(D,m) = \mathcal{B}(D),$$

where $\mathcal{B}(D)$ (resp. $\mathcal{B}_H(D)$) is Bloch space (a space of all harmonic functions of Bloch type) and $BMOA(D)$ (resp. $BMOH(D)$) is a usual BMO space, i.e., a space of all analytic (harmonic) functions whose boundary functions are BMO on ∂D. Therefore $BMO(D,m)$ and $BMOH(D,\lambda)$ are well-known spaces. On the other hand, it seems to be not well-known about these BMO properties for potentials on D. Here we investigate the fundamental property of potentials of BMO on $\mathbb{R}^n$, D, and Riemann surfaces in § 1, 2 and 3 respectively and give some remark on BMO spaces on plane domain in § 4.

§ 1. BMO property for potentials on $\mathbb{R}^n$.

Here we give the characterization of measures of Newtonian potentials (resp. logarithmic potentials in case n = 2) of BMO($\mathbb{R}^n$). The following theorem is fundamental.

Theorem 1. Let $0 < \alpha < n$ and Φ be a function on $\mathbb{R}^n$ such that (i) $\Phi(ax) = \Phi(x)$, $a > 0$, $x \in \mathbb{R}^n$ (ii) $|\Phi(x) - \Phi(y)| \leq K|x-y|$, $|x| = |y| = 1$. Let μ be a positive measure on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} |\Phi(x-y)|/|x-y|^{n-\alpha}\, d\mu(y) \in L^1_{loc}(\mathbb{R}^n) \quad \text{and} \quad \sup_B \mu(B)/(\mathrm{rad}(B))^{n-\alpha} < \infty,$$

where the supremum is taken for every ball B in $\mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} \Phi(x-y)/|x-y|^{n-\alpha}\, d\mu(y) \in \mathrm{BMO}(\mathbb{R}^n).$$

(proof) Let $B_{x_0,r} = \{x \in \mathbb{R}^n : |x-x_0| < r\}$. By direct calculation,

$$\left|\frac{\Phi(x-y)}{|x-y|^{n-\alpha}} - \frac{\Phi(x_0-y)}{|x_0-y|^{n-\alpha}}\right| \leq K_1 \frac{|x_0-x|}{|x_0-y|^{n-\alpha+1}}, \quad x \in B_{x_0,r}, \quad y \in \mathbb{R}^n \setminus B_{x_0,2r},$$

therefore when $x \in B_{x_0,r}$,

$$\int_{\mathbb{R}^n \setminus B_{x_0,2r}} \left|\frac{\Phi(x-y)}{|x-y|^{n-\alpha}} - \frac{\Phi(x_0-y)}{|x_0-y|^{n-\alpha}}\right| d\mu(y) \leq K_1 \int_{\mathbb{R}^n \setminus B_{x_0,2r}} \frac{|x_0-x|}{|x_0-y|^{n-\alpha+1}}\, d\mu(y)$$
$$\leq K_2 .$$

Similarly, we obtain

$$\int_{B_{x_0,r}} \int_{B_{x_0,2r}} \frac{|\Phi(x-y)|}{|x-y|^{n-\alpha}}\, d\mu(y)dx \leq K_3 r^n .$$

By these two inequalities above,

$$\int_{B_{x_0,r}} \left| \int_{\mathbb{R}^n} \frac{\Phi(x-y)}{|x-y|^{n-\alpha}}\, d\mu(y) - \int_{\mathbb{R}^n \setminus B_{x_0,2r}} \frac{\Phi(x_0-y)}{|x_0-y|^{n-\alpha}}\, d\mu(y) \right| dx \leq K_4 r^n.$$

Since $B_{x_0,r}$ is arbitrary, the proof is complete.

Corollary 1. Let μ be a positive measure on $\mathbb{R}^n$, $n \geq 3$, such that $N^{\mu}(x) = \int_{\mathbb{R}^n} 1/|x-y|^{n-2}\, d\mu(y) \in L^1_{loc}(\mathbb{R}^n)$. Then N^{μ} belongs to $BMO(\mathbb{R}^n)$ if and only if $\sup_B \mu(B)/(rad(B))^{n-2} < \infty$.

(proof) Let $\phi(x)$ be a non-negative C^{∞}-function on $\mathbb{R}^n$ such that (i) $\phi(x) = 1$, $|x| \leq 1$, (ii) $\phi(x) = 0$, $|x| \geqslant 2$. Since $\Delta N^{\mu} = -K\mu$ where $K > 0$ is a constant, we have

$$\mu(B_{x,r}) \leq \int \phi((y-x)/r)d\mu(y) = (-1/Kr^2)\int \Delta\phi((y-x)/r)N^{\mu}(y)dy$$
$$= (-1/Kr^2)\int \Delta\phi((y-x)/r)\left\{N^{\mu}(y)-N^{\mu}(B_{x,2r})\right\}dy$$
$$\leq (1/Kr^2)\,\|\Delta\phi\|_{\infty}\int_{B_{x,2r}} |N^{\mu}(y)-N^{\mu}(B_{x,2r})|\,dy$$
$$\leq (1/Kr^2)\,\|\Delta\phi\|_{\infty} K_1 r^n \,\|N^{\mu}\|_{BMO(\mathbb{R}^n)} \leq K_2\,\|N^{\mu}\|_{BMO(\mathbb{R}^n)}\, r^{n-2}$$

where $N^{\mu}(B_{x,2r})$ is the mean value of N^{μ} on $B_{x,2r}$ with respect to the n-dim. Lebesgue measure dy . Q.E.D.

Since $(\partial/\partial\bar{z})T^{\mu} = -\pi\mu$, and $\Delta G^{\mu} = -2\pi\mu$ for the Cauchy transform $T^{\mu}(z) = \int \frac{1}{\zeta - z}\, d\mu(\zeta)$ and the logarithmic potential $G^{\mu}(z) = \int \log(1/|z-\zeta|)d\mu(\zeta)$ of a measure μ on $\mathbb{R}^2$, we can prove the followings as above.

<u>Corollary 2</u>. Let μ be a positive measure on $\mathbb{R}^2$ such that $\int 1/|\zeta - z|\, d\mu(\zeta) \in L^1_{loc}(\mathbb{R}^2)$. Then $T^{\mu}(z)$ belongs to $BMO(\mathbb{R}^2)$ if and only if $\sup_B \mu(B)/(rad(B)) < \infty$.

<u>Corollary 3</u>. Let μ be a positive measure on $\mathbb{R}^2$ such that $\int |\log(1/|z-\zeta|)|\,d\mu(\zeta) \in L^1_{loc}(\mathbb{R}^2)$. Then $G^{\mu}(z)$ belongs to $BMO(\mathbb{R}^2)$ if and only if μ is a finite measure.

Using this result, we can prove the following estimate of $BMO(\mathbb{R}^2)$

norms of the Green functions on plane domain.

Corollary 4. Let Ω be a hyperbolic plane domain and $g_\Omega(z,\zeta)$ the Green function on Ω with pole $\zeta \in \Omega$. Then

$$C_1 \leq \| g_\Omega(z,\zeta)\|_{BMO(\mathbb{R}^2)} \leq C_2 ,$$

where we extend this function as 0 to $\mathbb{R}^2 \setminus \Omega$ and $C_1, C_2 > 0$ are universal constants.

§ 2. BMO property for potentials on the unit disk D.

We note that BMO(D,m), BMO(D,λ), BMO(∂D) are conformally invariant. We say a space Y of functions on D or on ∂D with norm $\| \|$ is conformally invariant when there exists a constant $K \geq 1$ such that for every $S \in$ Möb(D) and $f \in Y$,

$$\frac{1}{K}\|f\| \leq \|f\circ S\| \leq K\|f\| .$$

BMO(D,λ) is obviously conformally invariant with $K = 1$, on the other hand, the conformal invariance of other two BMO spaces is not trivial (see[9]). By this property, if the Green potential on D of measure μ belongs to BMO(D,m) or BMO(D,λ), then μ must have some conformally invariant property. Carleson measure has a conformally invariant property as below.

Proposition 1. For a positive measure μ on D, the following conditions are equivalent.

(1) $(1-|z|^2)d\mu(z)$ is a Carleson measure.

(2) $\sup\{\langle\nu,m\rangle : \nu \in \mathcal{M}(\mu)\} < \infty$.

(3) $\sup\{Q^\mu(z) : z \in D\} < \infty$.

(4) $\sup\{\int_D (1-|z|^2)d\nu(z) : \nu \in \mathcal{M}(\mu)\} < \infty$,

where $Q^{\mu}(z) = \int_D k(z,\zeta)d\mu(\zeta)$, $k(z,\zeta) = (1-|z|^2)(1-|\zeta|^2)/(|z-\zeta|^2)$

$$\mathcal{M}(\mu) = \{\nu : \nu = \mu S^{-1},\ S \in \text{M\"ob}(D)\} .$$

$$\langle \nu, m\rangle = \int_D\int_D g(z,\zeta)d\nu(z)dm(\zeta) .$$

For (1)$\leftrightarrow$(3), see [3] . And this proposition follows from the equation $\int_D g(z,\zeta)dm(\zeta) = \frac{\pi}{2}(1-|z|^2)$, $z \in D$. Our main result is

Theorem 2. For a positive measure μ on D, the following conditions are equivalent.

(1) $P^{\mu} \in BMO(\mathbb{R}^2)$ when we extend this function as 0 to $\mathbb{R}^2 \setminus D$.

(2) $(1-|z|^2)d\mu(z)$ is a Carleson measure,

moreover, if μ satisfies(1) (and(2)) then $P^{\mu} \in BMO(D,\lambda)$, where P^{μ} is the Green potential on D of measure μ .

We need the next proposition to prove this theorem.

Proposition 2 ([7]). Let f be a function in $L^1_{loc}(D)$ such that

$$\sup \frac{1}{m(B)}\int_B |f-f(B,m)|dm < \infty ,$$

where the supremum is taken for every disk B in D whose hyperbolic radius is less than some given constant. Then $f \in BMO(D,m)$. In this sence, BMO(D,m)-property is a local property.

We can restate this proposition as follows.

Proposition 2'. Let $0 < r_0 < 1$ and f be a function in $L^1_{loc}(D)$ such that

$$\sup\left\{\frac{1}{m(D_r)}\int_{D_r} |g-g(D_r,m)|dm : 0<r<r_0,\ g = f\circ S,\ S \in \text{M\"ob}(D)\right\} < \infty$$

where $D_r=\{|z|<r\}$. Then f belongs to BMO(D,m).

(proof of Theorem 2) First we prove (2) $\rightarrow$ (1). Let $(1-|z|^2)d\mu(z)$ be a Carleson measure. Fix r_1, r_2, $0 < r_1 < r_2 < 1$. By Proposition 1 (4),

$$\int_{\{r_2 < |\zeta| < 1\}} g(z,\zeta)\,d\nu(\zeta) \le K_1 , \quad |z| < r_1 , \qquad \nu \in \mathcal{M}(\mu) ,$$

$$\int_{\{|\zeta| \le r_2\}} d\nu(\zeta) \le K_2 , \qquad \nu \in \mathcal{M}(\mu) .$$

By the second inequality, we have

$$\left\| \int_{\{|\zeta| \le r_2\}} g(z,\zeta)\,d\nu(\zeta) \right\|_{BMO(D,m)} \le K_3 , \qquad \nu \in \mathcal{M}(\mu) .$$

Above inequalities and Proposition 2' imply $P^{\mu} \in BMO(D,m)$. And this proof shows that for a fixed constant $a_0 > 0$,

$$\sup\left\{ P^{\mu}(B,m) : \text{the hyperbolic radius of } B \text{ equals to } a_0 \right\} < \infty .$$

Therefore, we can show that

$$\sup\left\{ P^{\mu}(S,m) : S = \left\{ z = re^{i\theta} : 1-h < r < 1,\ \theta_0 < \theta < \theta_0 + h \right\} \right\} < \infty ,$$

where $P^{\mu}(S,m)$ is the mean value of P^{μ} on the set S with respect to dm. Hence it is easy to prove $P^{\mu} \in BMO(\mathbb{R}^2)$.

Next we prove (1) $\rightarrow$ (2). Let μ satisfies (1) of this theorem. Since $BMO(\mathbb{R}^2)$ is invariant under $Möb(\mathbb{R}^2)$ (see[9]), there exists a constant $K_4 > 0$ such that $\|P^{\nu}\|_{BMO(\mathbb{R}^2)} \le K_4$, $\nu \in \mathcal{M}(\mu)$, where we extend P^{ν} as 0 to $\mathbb{R}^2 \setminus D$. Therefore, if $\nu \in \mathcal{M}(\mu)$ then

$$\begin{aligned} K_4 &\geq \frac{1}{m(D_2)} \int_{D_2} |P^{\nu}(z) - P^{\nu}(D_2,m)|\,dm \\ &= \frac{1}{4\pi} \int_D \left|P^{\nu}(z) - \frac{1}{4}P^{\nu}(D,m)\right| dm(z) + \frac{1}{4\pi} \int_{D_2 \setminus D} \frac{1}{4} P^{\nu}(D,m)\,dm(z) \\ &\geq 0 + \frac{1}{4} \cdot 3\pi \cdot \frac{1}{4} P^{\nu}(D,m) = \frac{3}{16} P^{\nu}(D,m) , \end{aligned}$$

therefore $\int_D P^{\nu}(z)\,dm(z) = \pi P^{\nu}(D,m) \le \frac{16\pi}{3} K_4$. And (1) $\rightarrow$ (2) follows from Proposition 1.

We now prove the last statement of this theorem. Since dm and $d\lambda$ are comparable on $\{|z| \leq 1/2\}$, BMO(D,m) property implies that

$$\frac{1}{\lambda(D_r)}\int_{D_r} |P^{\nu}(z)-P^{\nu}(D_r,\lambda)|\,d\lambda(z) \leq K_5 , \quad \nu \in \mathcal{M}(\mu) , \quad 0 < r \leq 1/2 .$$

Further if $1/2 < r < 1$ and $\nu \in \mathcal{M}(\mu)$ then

$$\begin{aligned}\frac{1}{\lambda(D_r)}\int_{D_r} |P^{\nu}(z)- 0\ |\,d\lambda(z) &\leq \frac{1}{m(D_r)}\int_{D_r} P^{\nu}(z)\,dm(z)\\ &\leq \frac{4}{\pi}\int_{D} P^{\nu}(z)\,dm(z) \leq K_6 ,\end{aligned}$$

and so $P^{\mu} \in$ BMO(D,λ). Q.E.D.

Since $\langle \nu ,m\rangle \leq \langle m,m\rangle^{\frac{1}{2}}\langle \nu , \nu\rangle^{\frac{1}{2}}$ for a positive measure ν on D,

<u>Corollary 5</u>. Every potential on D with finite energy belongs to BMO(D,λ).

If $\{z_n\}_{n=1}^{\infty}$ is a interpolating sequence on D, then $\sum_n (1-|z_n|^2)d\,\delta_{z_n}$ is a Carleson measure, where δ_{z_n} is Dirac measure at z_n , hence

<u>Corollary 6</u>. If $\{z_n\}_{n=1}^{\infty}$ is a interpolating sequence on D, then

$$\sum_{n=1}^{\infty} g(z,z_n) \in \mathrm{BMO}(D,\lambda) .$$

On the other hand, there exists a positive measure μ on D such that $P^{\mu} \in$ BMO(D,λ) and μ does not satisfy the condition of Theorem 2.

<u>Example</u>. Let $f(z) = \log(1-|z|^2)$, $f_r(z) = f(rz)-f(r)$, $0<r<1$, then f_r, $0<r<1$, are potentials and we can show that f , f_r , $0<r<1$, belong to BMO(D,λ) and their BMO(D,λ) norms are bounded

above. Let r_n , $0 < r_n < 1$ be a sequence which tends to 1. Since $\int_D f_{r_n} dm \to \infty$, $n \to \infty$, we can choose sequences $\{\alpha_n\}_{n=1}^{\infty}$, $\alpha_n > 0$ and $\{A_n\}_{n=1}^{\infty}$, $A_n \in$ Möb(D), such that $\sum_n \alpha_n (f_{r_n} \circ A_n)$ is a potential of BMO(D,λ) which does not satisfy the condition of Theorem 2 .

The necessary and sufficient condition for potential to be in BMO(D,m) is

<u>Theorem 3</u>. For a positive measure μ on D, its potential belongs to BMO(D,m) if and only if μ satisfies the following two conditions.

(1) μ is uniformly locally finite.

(2) $\sup_{\nu} \left| \int_D (1-|z|^2)\cos\theta \, d\nu(z) \right| < \infty$, $z = re^{i\theta}$,

where the supremum is taken for every $\nu \in \mathcal{M}(\mu)$, and we say a positive measure μ on D is uniformly locally finite if

$\sup \left\{ \mu(B) : \text{hyperbolic radius of } B \text{ is less than } a_0 \right\} < \infty$,

for some constant $a_0 > 0$.

See [5] for the detail. Compare this condition with the condition (4) of Proposition 1.

For the last of this section, we study whether Riesz decomposition of superharmonic function preserve these BMO properties or not.

<u>Theorem 4</u>. Let h be a harmonic function on D and μ be a positive measure on D such that $s = h + P^{\mu}$ belongs to BMO(D,λ), then both h and P^{μ} belong to BMO(D,λ). That is, Riesz decomposition preserve BMO(D,λ) property.

The proof of this theorem is almost the same as the proof of the relation $BMO(D,\lambda) = BMOH(D)$ (see [4]). On the other hand, Riesz decomposition does not preserve $BMO(D,m)$ property. Indeed we can construct a positive superharmonic function of $BMO(D,m)$ whose harmonic part does not belongs to $BMOH(D,m)$ using the function $\log(1-|z|^2)$ (see [5]).

§ 3. BMO property for potentials on Riemann surfaces.

Let R be a hyperbolic Riemann surface and $\pi : D \to R$ be its universal covering map. We can define the space $BMO(R,m)$ by

$$BMO(R,m) = \left\{ f : f\circ\pi \in BMO(D,m) \right\} .$$

$BMOH(R,m)$, $BMOA(R,\lambda)$, etc., are defined as the same way. The following result is known about BMO property for potentials.

Proposition 3 ([7]). Potentials of positive measure with compact supports belong to $BMO(R,\lambda)$.

We shall extend this result. Let Γ be the covering transformation group associated with $\pi : D \to R$. We define a function k_R on $R \times R$ by

$$k_R(\pi(z), \pi(\zeta)) = \sum_{A\in\Gamma} k(z, A\zeta) , \qquad z, \zeta \in D ,$$

where k is the function defined in Proposition 1 . Since $k(z,\zeta)$ is conformally invariant, k_R is well-defined. Our main theorem is

Theorem 5. Let μ be a positive measure on R such that

$$\sup_{p\in R} \int_R k_R(p,q)\, d\mu(q) < \infty \qquad ,$$

then its Green potential P_R^{μ} belongs to $BMO(R,\lambda)$.

This theorem is the consequence of Theorem 2 and the following lemma which we can prove by direct calculation.

Lemma 1. Let μ be a positive measure on R and μ_D be a positive measure on D such that $P^{\mu_D} = P_R^{\mu} \circ \pi$, then

$$\sup_{z \in D} \int_D k(z,\zeta)d\mu_D(\zeta) = \sup_{p \in R} \int_R k_R(p,q)d\mu(q) .$$

For the boundedness of k_R , we have

Lemma 2. k_R is bounded above if and only if there exists a constant $K > 0$ such that for every $q \in R$, the domain

$$\{ p \in R : g_R(p,q) > K \}$$

is simply connected, where g_R is the Green function on R. Especially , compact bordered Riemann surfaces satisfy this condition.

Corollary 7. Let R be a Riemann surface which satisfy the condition of Lemma 2, then potentials on R of positive finite measures belong to BMO(R,λ). Especially, compact bordered surfaces have this property.

On the other hand, there exists a Riemann surface R and a positive finite measure μ on R such that its potential P_R^{μ} does not belongs to BMO(D,m). Indeed, we can construct such a measure on $R = \{ 0 < |z| < 1 \}$.

Also, we can extend Corollary 5 as follows.

Theorem 6. Let R be a Riemann surface which satisfy the condition of Lemma 2, then potentials on R of finite energy belong to BMO(R,λ).

(proof) For a positive measure μ on R, we can show the following inequality by direct calculation.

$$\sup \langle \nu, m \rangle \leq \frac{\pi}{\sqrt{2}} \langle \mu, \mu \rangle_R^{\frac{1}{2}} \|k_R\|_\infty^{\frac{1}{2}}$$

where $\langle \mu, \mu \rangle_R$ is the energy integral on R and μ_D be a measure on D such that $P^{\mu_D} = P_R^{\mu} \circ \pi$ and the supremum is taken for every $\nu \in \mathcal{M}(\mu_D)$. Hence, this theorem follows from Theorem 2.

§ 4. Remarks on BMO spaces on plane domain.

Let Ω be a hyperbolic plane domain. In this case, we can define the following BMO space with respect to the 2-dim. Lebesgue measure $dm = dxdy$ on Ω.

$$\widetilde{BMO}(\Omega, m) = \left\{ f \in L^1_{loc}(\Omega) : \sup_B \frac{1}{m(B)} \int_B |f - f(B,m)|\, dm < \infty \right\},$$

where $f(B,m) = \frac{1}{m(B)} \int_B f\, dm$ and the supremum is taken for every disk B on Ω.

$$\widetilde{BMOH}(\Omega, m) = \widetilde{BMO}(\Omega, m) \cap H(\Omega), \quad \widetilde{BMOA}(\Omega, m) = \widetilde{BMO}(\Omega, m) \cap A(\Omega).$$

The following characterization is known.

Proposition 4 ([1]). For a harmonic function h on Ω,

(1) $h \in \widetilde{BMOH}(\Omega, m)$ if and only if there exists a constant $K > 0$ such that $|\nabla h(z)| \leq K/d(z, \partial\Omega)$, $z \in \Omega$.

(2) $h \in BMOH(\Omega, m)$ if and only if there exists a constant $K > 0$ such that $|\nabla h(z)| \leq K \rho_\Omega(z)$, $z \in \Omega$,

where $d(z, \partial\Omega)$ is the distance between z and $\partial\Omega$, and $\rho_\Omega(z)|dz|$ is the hyperbolic metric on Ω.

Since $\rho_\Omega(z) \leq 2/d(z, \partial\Omega)$, we have $BMOH(\Omega, m) \subset \widetilde{BMOH}(\Omega, m)$. Further we obtain $HD(\Omega) \subset \widetilde{BMOH}(\Omega, m)$ as a corollary of this proposition. On the other hand, $HD(\Omega) \not\subset BMOH(\Omega, m)$ in general (see [4]).

Reimann[8] and Jones[6] have proved the following remarkable result.

Proposition 5. Let Ω_1 and Ω_2 are plane domain, $f: \Omega_1 \to \Omega_2$ be a conformal map and h be a function in $\widetilde{BMO}(\Omega_2,m)$, then $h\circ f$ belongs to $\widetilde{BMO}(\Omega_1,m)$ and

$$\|h\circ f\|_{\widetilde{BMO}(\Omega_1,m)} \leqslant C \|h\|_{\widetilde{BMO}(\Omega_2,m)} ,$$

where C is a universal constant.

Our main result in this section is

Theorem 7. Let Ω be a hyperbolic plane domain with universal covering $\pi: D \to \Omega$. Then $BMO(\Omega,m) \subset \widetilde{BMO}(\Omega,m)$, further the following conditions are equivalent.

(1) $BMO(\Omega,m) = \widetilde{BMO}(\Omega,m)$.

(2) $BMOH(\Omega,m) = \widetilde{BMOH}(\Omega,m)$.

(3) $1/d(z,\partial\Omega) \leqslant A\, \rho_\Omega(z)$, $z \in \Omega$.

(4) There exists a constant $K > 0$ such that for every $z_0 \in \Omega$, the domain $\{z \in \Omega : \rho_\Omega(z,z_0) < K\}$ is simply connected.

(5) $\log \pi'(z) \in BMOA(D,m)$ $(= \mathcal{B}(D))$.

(6) $\log \rho_\Omega(z) \in BMO(\Omega,m)$,

where $\rho_\Omega(z,z_0)$ is the hyperbolic distance between z and z_0 .

(proof) The inclusion $BMO(\Omega,m) \subset \widetilde{BMO}(\Omega,m)$ is the consequence of Proposition 5. "(1) $\to$ (2)" is trivial.

((2) $\to$ (3)) If Ω does not satisfy the condition (3), then there exists sequences $\{z_n\}_{n=1}^{\infty}$ in Ω and $\{\zeta_n\}_{n=1}^{\infty}$ on $\partial\Omega$ such that $1/|z_n - \zeta_n| \geqslant n\, \rho_\Omega(z_n)$. Set $u_n(z) = \log|z-\zeta_n|$ then

$$\|u_n\|_{BMOH(\Omega,m)} \leqslant K_1 , \quad n = 1, 2, \ldots..$$

because $\log|z| \in BMO(\mathbb{R}^2)$. On the other hand,

$$\|u_n\|_{BMOH(\Omega,m)} \geqslant K_2 \sup_{z\in\Omega} |\nabla u_n(z)| \frac{1}{\rho_\Omega(z)} \geqslant K_2 |\nabla u_n(z_n)| \frac{1}{\rho_\Omega(z_n)}$$
$$\geqslant K_2(1/|z_n-\zeta_n|)n|z_n-\zeta_n| = K_2 n \to \infty , \quad n \to \infty ,$$

hence we can construct a harmonic function of $\widetilde{BMOH}(\Omega,m) \setminus BMOH(\Omega,m)$ by taking some infinite linear combination of u_n.

((3) $\to$ (4)) We will show that if we choose the constant K such that $K > \pi/A$ then (4) is valid with this K. If there exists a point $z_0 \in \Omega$ such that $\Omega_0 = \{z \in \Omega : \rho_\Omega(z,z_0) < K\}$ is not simply connected, then there exists a closed couve C in Ω_0 such that $\int_C 1/d(\zeta,\partial\Omega)|d\zeta| < 2\pi$ and C surrounds some point $\zeta_0 \in \partial\Omega$.
Hence,

$$\int_C \frac{|d\zeta|}{d(\zeta,\partial\Omega)} \geqslant \int_C \frac{d\zeta}{|\zeta-\zeta_0|} \geqslant \int_C \frac{1}{r} r\,|d\theta| \geqslant 2\pi ,$$

where $\zeta - \zeta_0 = re^{i\theta}$. This is a contradiction.

"(4) $\to$ (1)" is the consequence of Proposition 5 and Proposition 2.

"(4) $\leftrightarrow$ (5)" is well-known result. Since $1/(1-|z|^2) = \rho_\Omega(\pi(z))|\pi'(z)|$, "(5) $\leftrightarrow$ (6)" follows from the fact $\log(1-|z|^2) \in BMO(D,m)$.

Q.E.D.

References

[1] J. A. Cima and I. Graham, Removabilities for Bloch and BMO functions, Illinois J. Math. 27 (1983), 691-703.

[2] R. R. Coifman, R. Rochberg and G. Weiss, Factorization theorems for Hardy spaces in several complex variables, Ann. of Math. 103, (1976), 611-635.

[3] J. B. Garnett, "Bounded analytic functions", Academic Press 1981.

[4] Y. Gotoh, On BMO functions on Riemann surface, J. Math. Kyoto Univ. 25 (1985), 331-339.

[5] ________, On BMO property for potentials, in preparation.

[6] P. Jones, Extension theorems for BMO, Indianna Univ. Math. J. 29, (1979), 41-66.

[7] Y. Kusunoki and M. Taniguchi, Remarks on functions of bounded mean oscillation on Riemann surface, Kodai Math. J. 6 (1983) 434-442.

[8] H. M. Reimann, Functions of bounded mean oscillation and quasiconformal mappings, Comm. Math. Helv. 49 (1974), 260-276.

[9] H. M. Reimann and T. Rychener, "Funktionen beschränkter mittelerer Oszillation", Lecture Notes in Math. 489 Springer 1975.